Qualifying Exam (January 2017): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.

Do 2 out of problems 1,2,3.

Do 2 out of problems 4,5,6.

Do 3 out of problems 7,8,9,10,11,12,13,14.

All problems are weighted equally. **On this cover page write which seven problems you want graded.**

problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

Signature

(1). The following LP was solved (using the big M method) and the optimal tableau is given below. e_1 and e_2 are the excess variables subtracted from the first and second constraints, and a_i is the artificial variable of the ith constraint.

$$\begin{array}{lll} \max & z = 4x_1 + x_2 & \\ \text{s.t.} & 3x_1 + x_2 & \geq 6 \\ & 2x_1 + x_2 & \geq 4 \\ & x_1 + x_2 & = 3 \\ & x_1, x_2 & \geq 0 \end{array}$$

z	x_1	x_2	e_1	e_2	a_1	a_2	a_3	rhs
1	0	3	0	0	M	M	$M+4$	12
0	1	1	0	0	0	0	1	3
0	0	2	1	0	-1	0	3	3
0	0	1	0	1	0	-1	2	2

(a). Find the dual of this LP and its optimal solution (the objective value and the value of the dual variables). Use the tableau - do not solve from scratch!

(b). Find the range of values of the objective function coefficient for x_2 for which the current basis remains optimal.

(c). Find the range of values of b_2 for which the current basis remains optimal.

(d). We wish to add to the LP the constraint $x_2 \geq 1$, for which the current optimal solution is not feasible. Set up a tableau on which to proceed by the dual simplex method to find the new optimal solution.

(e). Solve the problem set up in part (d) using the dual simplex method. Note, if you are doing more than 2 pivots, something is wrong!

(2). Consider an upper bounded transshipment problem and a feasible tree solution:

(a). Suppose the current feasible tree solution is degenerate. Must the next feasible tree solution (obtained using the algorithm from class) also be degenerate? If so give a short proof. If not, give a counterexample.

(b). Suppose the current feasible tree solution is nondegenerate. Must the next feasible tree solution also be nondegenerate? If so give a short proof. If not, give a counterexample.

(c). Suppose the current feasible tree solution that is nondegenerate. Further suppose that x_{ij} enters the tree, and when calculating t, the change in the flow, a unique (single) arc in the cycle achieves the minimum. Prove that the next feasible tree solution is also non-degenerate.

(3). Consider an LP $\min\{z = c^t x \mid Ax = b,\ x \geq 0\}$ being solved by a modified version of the two-phase method, in which the first phase is $\min\{w' = \sum_j j a_j \mid Ax + Ia = b,\ x \geq 0, a \geq 0\}$. (Reminder, the phase 1 described in class is $\min\{w = \sum_j a_j \mid Ax + Ia = b,\ x \geq 0, a \geq 0\}$.) Let w'^* be the optimal solution to the modified phase 1 problem, and let w^* be the optimal solution to the phase 1 problem from class.

(a). Prove that if $w'^* = 0$ then the original LP is feasible.

(b). Prove that if $w'^* > 0$ then the original LP is not feasible.

(c). Prove that $w'^* \geq w^*$.

(d). Parts (a) and (b) show that the two-phase method can be used with a different objective function for phase 1. Can we use any arbitrary cost vector $d = (d_1, d_2, \ldots, d_n)$? In other words, for any d can we define phase 1 as $\min\{w(d) = \sum_j d_j a_j \mid Ax + Ia = b,\ x \geq 0, a \geq 0\}$, and conclude that the original LP is feasible if and only if this phase 1 optimum solution is $w^*(d) = 0$? Give a short proof or counterexample.

(4). Let $\{N(k), k = 0, 1, \ldots\}$ be a discrete-time renewal process and $\{X_i, i \geq 1\}$ be the sequence of i.i.d. inter-arrival times with common probability mass function $f_j = P(X_i = j)$ for all i. Let $\{B_n\}$ be the excess life process associated with $\{N(k)\}$, i.e., $B_n = S_{N(n)+1} - n$, where S_n is the time of the occurrence of the nth event. Show that $\{B_n\}$ forms a Markov chain (write down its transition matrix) and compute the long run steady state distribution of the chain.

(5). You arrive at a bank with two tellers and find that both tellers are busy, each serving a customer. The service times of the tellers are independent exponential random variables with parameters λ and μ, respectively. Suppose you randomly pick (with equal probability) a teller and join the service as soon as that teller becomes free. What is the probability you are the last of the three customers to leave the bank?

(6). Customers arrive at a service facility according to a Poisson process $PP(2)$. There are two identical servers at the facility, where the service times at each server are i.i.d. exponential random variables with parameter 1. A customer will only enter the facility when at least one server is free. When both servers are free, an arriving customer is equally likely to be served by either one. Suppose both servers are free at time 0, what is the probability that both servers are busy at time t? What is the steady state probability that both servers are busy?

(7). (a). Define "zone" and state the Zone Theorem for an arrangement of lines in the plane.

(b). Sketch a proof of the Zone Theorem.

(c). Explain how the Zone Theorem enables an efficient algorithm to construct the DCEL of an arrangement of n lines in the plane. What is the running time of the algorithm? Explain.

(8). Let P be a polygonal domain in the plane, having a total of n vertices and h holes.

(a). Describe an efficient algorithm for determining the visibility polygon, $VP(p)$, of a point $p \in P$. (Recall that $VP(p)$ is the set of all points of P that are visible to p within P.) What is the running time (in big-Oh) for the algorithm?

(b). Now suppose we want to preprocess P to support motion planning queries of the following form: Given a starting point $s \in P$ and a destination point $t \in P$, compute a polygonal path within P from s to t (for a "point robot", which we can think of as a zero-radius disk). What preprocessing would you do for P? What is its efficiency? How efficiently can a query (s, t) be answered?

(9). Consider an MDP with finite state and action sets. The goal is to maximize

$$E_x^\pi \sum_{t=0}^{\infty} \beta_1^t r_1(x_t, a_t) + E_x^\pi \sum_{t=0}^{\infty} \beta_2^t r_2(x_t, a_t),$$

where $\beta_i \in (0, 1)$ are constants and r_i are one-step reward functions with finite values, $i = 1, 2$. Prove that there are optimal nonrandomized Markov policies for this MDP.

(10). For the MDP described in Problem 9, provide an example for which a stationary optimal policies does not exist.

(11). Let $X_1, \ldots, X_n$ be a sequence of i.i.d. random variables uniformly distributed over $[0, 1]$. Let $X_{(k)}$ be the kth order statistic of the sequence. Find the density function of $X_{(k)}$ and use this result to provide an algorithm for generating random variates from the density $f(x) = 6x(1 - x),\ 0 \leq x \leq 1$.

(12). Let U be a uniform random variable over $[0, 1]$. Let $X = 1 - U^2$. Find the density function $g(x)$ of X. Use $g(x)$ to construct a majorizing function and provide an acceptance-rejection algorithm for generating random variates from the density $f(y) = \frac{2}{\pi\sqrt{1-y^2}},\ 0 \leq y \leq 1$.

(13). Let $X_n \sim \exp(\lambda_n)$ be exponentially distributed random variables with the intensities λ_n, where λ_n are positive numbers such that the sequence $\{\lambda_n\}_{n=1,2,...}$ converges to a finite positive number λ. Is it true that $X_n \xrightarrow{\mathcal{D}} X$, where X is the exponentially distributed random variable with the intensity λ? Explain your answer.

(14). Let $\{P_n\}_{n=1,2,...}$ be a sequence of probabilities defined on a measuable space $(\Omega, \mathcal{F})$, and $\{\alpha_n\}_{n=1,2,...}$ be a sequence of nonnegative constants such that $\sum_{n=1}^{\infty} \alpha_n = 1$. For each $A \in \mathcal{F}$ define $P(A) := \sum_{n=1}^{\infty} \alpha_n P_n(A)$. Prove that P is a probability on $(\Omega, \mathcal{F})$ and, if $\alpha_n > 0$, then $P_n \ll P$.